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Approximation by modified Jain-Baskakov-Stancuo operators

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Abstract

In this paper, we introduce a Stancu type generalization of modified Jain-Baskakov operators with parameter c. We studied some direct results in ordinary approximation. Also, the rate of convergence in terms of the modulus of continuity and weighted approximation by these operators are studied. Lastly, we give better estimations of the above operators using King type approach.

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1 Introduction

In 2014, Patel and Mishra [32] introduced modified Jain-Baskakov operators as follows

$$\mathcal{K}_{n,c}^{\beta}(f;x) = \frac{(n-c)}{c}\sum_{v=1}^{\infty}\omega_{\beta}(v,nx)\int_0^{\infty}p_{n,v-1,c}(t)f(t)dt + e^{-nx}f(0), \tag{1.1}$$

where

$$p_{n,v-1,c}(t) = c\frac{\Gamma(n/c+v-1)}{\Gamma(v)\Gamma(n/c)}.\frac{(ct)^{v-1}}{(1+ct)^{n/c+v-1}} \tag{1.2}$$

and

$$\omega_{\beta}(v,nx) = nx(nx+v\beta)^{v-1}\frac{e^{-(nx+v\beta)}}{v!}, \tag{1.3}$$

and $\beta \in [0,1), f \in C[0,\infty)$. Some approximation properties of these operators were given in [14]. As a special case, i.e., $c = 1$, the operators (1.1) reduced in Jain-Baskakov operators which is defined in [32].

In [28], Stancu introduced the positive linear operators $P_n^{(\alpha,\gamma)} : C[0,1] \to C[0,1]$ by modifying the Bernstein polynomial as

$$P_n^{(\alpha,\gamma)}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k+\alpha}{n+\gamma}\right),$$

where $b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k},\ x \in [0,1]$ is the Bernstein basis function and α, γ are any two real numbers which satisfy the condition that $0 \leq \alpha \leq \gamma$.

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Motivated by his work, the Stancu type modification of many sequences of linear positive operators has been considered and studied (see [1], [7], [15], [17] [32], [34] etc.).
For $f \in C[0,\infty)$, $0 \le \alpha \le \gamma$ we introduce the following Stancu type generalization of the operators (1.1):

$$\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(f;x) = \frac{(n-c)}{n}\sum_{v=1}^{\infty}\omega_\beta(v,nx)\int_0^\infty p_{n,v-1,c}(t) f\left(\frac{nt+\alpha}{n+\gamma}\right)dt + e^{-nx} f\left(\frac{\alpha}{n+\gamma}\right) \quad (1.4)$$

For $\alpha=\gamma=0$, we denote $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(f;x)$ by $\mathcal{K}^{\beta}_{n,c}(f;x)$.
The aim of this paper is to study the basic convergence theorem, Voronovskaja type asymptotic result, rate of convergence, weighted approximation and pointwise estimation of the operators (1.4). Further, to obtain better approximation, we also propose modification of the operators (1.4) using King type approach.

2 Moment estimates

Lemma 2.1. [14] For $n > 3c$, we have

1. $\mathcal{K}^{\beta}_{n,c}(1;x) = 1$;
2. $\mathcal{K}^{\beta}_{n,c}(t;x) = \dfrac{nx}{(n-2c)(1-\beta)}$;
3. $\mathcal{K}^{\beta}_{n,c}(t^2;x) = \dfrac{n^2}{(n-2c)(n-3c)}\left[\dfrac{x^2}{(1-\beta)^2} + \dfrac{x(2-2\beta+\beta^2)}{n(1-\beta)^3}\right]$.

Lemma 2.2. For the operators $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(f;x)$ as defined in (1.4), the following equalities holds for $n > 3c$

1. $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(1;x) = 1$;
2. $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(t;x) = \dfrac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)}$;
3. $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(t^2;x) = \left\{\dfrac{n^4}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta)^2}\right\}x^2 + \left\{\dfrac{n^3+2\alpha n^2(n-3c)(1-\beta)^2}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta)^3}\right\}x + \dfrac{\alpha^2}{(n+\gamma)^2}$.

Proof. For $x \in [0,\infty)$, in view of Lemma 2.1, we have

$$\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(1;x) = 1.$$

Next, for $f(t)=t$, again applying Lemma 2.1, we get

$$\begin{aligned}\mathcal{K}^{\alpha,\gamma}_{n,c,\beta}(f;x) &= \frac{(n-c)}{n}\sum_{v=1}^{\infty}\omega_\beta(v,nx)\int_0^\infty p_{n,v-1,c}(t)\left(\frac{nt+\alpha}{n+\gamma}\right)dt + e^{-nx}\left(\frac{\alpha}{n+\gamma}\right)\\ &= \frac{n}{n+\gamma}\mathcal{K}^{\beta}_{n,c}(t,x) + \frac{\alpha}{n+\gamma} = \frac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)}.\end{aligned}$$

Proceeding similarly, we have

$$\begin{aligned}\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) &= \frac{(n-c)}{n}\sum_{v=1}^{\infty}\omega_\beta(v,nx)\int_0^\infty p_{n,v-1,c}(t)\left(\frac{nt+\alpha}{n+\gamma}\right)^2 dt + e^{-nx}\left(\frac{\alpha}{n+\gamma}\right)^2\\ &= \left(\frac{n}{n+\gamma}\right)^2\mathcal{K}_{n,c}^{\beta}(t^2,x) + \frac{2n\alpha}{(n+\gamma)^2}\mathcal{K}_{n,c}^{\beta}(t,x) + \left(\frac{\alpha}{n+\gamma}\right)^2\\ &= \left\{\frac{n^4}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta)^2}\right\}x^2 + \left\{\frac{n^3+2\alpha n^2(n-3c)(1-\beta)^2}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta)^3}\right\}x\\ &\quad+\frac{\alpha^2}{(n+\gamma)^2}.\end{aligned}$$

Q.E.D.

Lemma 2.3. For $f \in C_B[0,\infty)$ (space of all bounded and continuous functions on $[0,\infty)$ endowed with norm $\| f \| = \sup\{|f(x)| : x \in [0,\infty)\}$), $\| \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) \| \leq \| f \|$.

Proof. In view of (1.4) and Lemma 2.2, the proof of this lemma easily follows. Q.E.D.

Remark 2.4. For $n > 3c$, we have

$$\begin{aligned}\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}\left((t-x);x\right) &= \left(\frac{(n+\gamma)\left(n\beta+2c(1-\beta)\right)-n\gamma}{(n+\gamma)(n-2c)(1-\beta)}\right)x + \frac{\alpha}{n+\gamma},\\ &= \mu_{n,c,\beta}^{\alpha,\gamma}(x), \text{(say)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}\left((t-x)^2;x\right) &= \left\{\frac{n^4+(n-3c)(n+\gamma)(1-\beta)((n-2c)(n+\gamma)(1-\beta)-2n^2)}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta)^2}\right\}x^2\\ &\quad+\left\{\frac{n^3+2\alpha(1-\beta)^2(n-3c)(n^2-(n-2c)(n+\gamma)(1-\beta))}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta)^3}\right\}x\\ &\quad+\frac{\alpha^2}{(n+\gamma)^2},\\ &= \xi_{n,c,\beta}^{\alpha,\gamma}(x), \text{(say)}.\end{aligned}$$

3 Main results

Theorem 1. (Voronovskaja type theorem) Let $b > 0$ and $\beta_n \in (0,1)$ such that $n\beta_n \to l \in R$ as $n \to \infty$. Then for every $f \in C[0,b]$, f', f'' exists at a fixed point $x \in (0,b)$, we have

$$\lim_{n\to\infty} n\left(\mathcal{K}_{n,c,\beta_n}^{\alpha,\gamma}(f;x) - f(x)\right) = (\alpha + (\gamma + l + 2c)x)f'(x) + \frac{x(2(1+\alpha)+cx)}{2}f''(x).$$

Proof. Let $x \in (0,b)$ be fixed. From the Taylor's theorem, we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{3.1}$$

where $r(t,x)$ is the peano form of the remainder and $\lim_{t\to x} r(t,x)=0$.
Applying $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f,x)$ on both sides of (3.1), we have

$$\begin{aligned} n\left(\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f;x)-f(x)\right) &= nf'(x)\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}((t-x);x)+\frac{1}{2}nf''(x)\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}\left((t-x)^2;x\right) \\ &+n\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}\left((t-x)^2 r(t,x);x\right). \end{aligned}$$

In view of Remark 2.4, we have

$$\lim_{n\to\infty} n\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}((t-x);x)=\alpha+(\gamma+l+2c)x \tag{3.2}$$

and

$$\lim_{n\to\infty} n\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}\left((t-x)^2;x\right)=x(2(1+\alpha)+cx). \tag{3.3}$$

Now, we shall show that

$$\lim_{n\to\infty} n\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}\left(r(t,x)(t-x)^2;x\right)=0$$

By using Cauchy-Schwarz inequality, we have

$$\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}\left(r(t,x)(t-x)^2;x\right) \le \left(\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(r^2(t,x);x)\right)^{1/2}\left(\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}((t-x)^4;x)\right)^{1/2}. \tag{3.4}$$

We observe that $r^2(x,x)=0$ and $r^2(.,x)\in C[0,b]$. Then, it follows that

$$\lim_{n\to\infty} \mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(r^2(t,x);x)=r^2(x,x)=0, \tag{3.5}$$

in view of fact that $\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}((t-x)^4;x)=O\left(\frac{1}{n^2}\right)$.
Now, from (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} n\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}\left(r(t,x)(t-x)^2;x\right)=0. \tag{3.6}$$

From (3.2), (3.3) and (3.6), we get the required result. Q.E.D.

3.1 Local approximation

For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\| f-g \| +\delta \| g'' \|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g',g'' \in C_B[0,\infty)\}$. By, p. 177, Theorem 2.4 in [2], there exists an absolute constant $M>0$ such that

$$K_2(f,\delta)\le M\omega_2(f,\sqrt{\delta}), \tag{3.7}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} | f(x+h) - f(x) |,$$

we denote the first order modulus of continuity of $f \in C_B[0,\infty)$.

Theorem 2. Let $f \in C_B[0,\infty)$ and $n > 3c$, we have

$$| \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x) | \leq M\omega_2\left(f, \zeta_{n,c,\beta}^{\alpha,\gamma}(x)\right) + \omega\left(f, \mu_{n,c,\beta}^{\alpha,\gamma}(x)\right),$$

where M is a positive constant and

$$\zeta_{n,c,\beta}^{\alpha,\gamma}(x) = \left(\xi_{n,c,\beta}^{\alpha,\gamma}(x) + \left(\mu_{n,c,\beta}^{\alpha,\gamma}(x)\right)^2\right)^{1/2}.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}$ defined by

$$\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x) = \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f\left(\frac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)}\right) + f(x). \tag{3.8}$$

From Lemma 2.2, we observe that the operators $\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}$ are linear and reproduce the linear functions. Hence

$$\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}((t-x);x) = 0. \tag{3.9}$$

Let $g \in W^2$ and $x,t \in [0,\infty)$. By Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}$ on both sides of the above equation and using (3.9), we get

$$\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(g;x) - g(x) = \overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}\left(\int_x^t (t-v)g''(v)dv;x\right).$$

Thus, by (3.8) we get

$$\begin{aligned}
|\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(g;x) - g(x)| &\leq \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}\left(\left|\int_x^t (t-v)g^{''}(v)dv\right|;x\right) \\
&\quad + \left|\int_x^{\frac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)}} \left(\frac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)} - v\right)g^{''}(v)dv\right| \\
&\leq \left(\xi_{n,c,\beta}^{\alpha,\gamma}(x) + \left(\mu_{n,c,\beta}^{\alpha,\gamma}(x)\right)^2\right) \parallel g^{''} \parallel \\
&\leq \left(\zeta_{n,c,\beta}^{\alpha,\gamma}(x)\right)^2 \parallel g^{''} \parallel .
\end{aligned} \tag{3.10}$$

On other hand, by (3.8) and Lemma 2.3, we have

$$|\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x)| \quad \leq \quad \| f \| . \tag{3.11}$$

Using (3.10) and (3.11) in (3.8), we obtain
$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x)|$

$$\begin{aligned}
&\leq |\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f-g;x)| + |(f-g)(x)| + |\overline{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(g;x) - g(x)| \\
&\quad + \left| f\left(\frac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)}\right) - f(x)\right| \\
&\leq 2 \| f-g \| + \left(\zeta_{n,c,\beta}^{\alpha,\gamma}(x)\right)^2 \| g'' \| + \left| f\left(\frac{n^2x+\alpha(n-2c)(1-\beta)}{(n+\gamma)(n-2c)(1-\beta)}\right) - f(x)\right|.
\end{aligned}$$

Taking infimum over all $g \in W^2$, we get

$$| \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x) | \quad \leq \quad K_2\left(f, (\zeta_{n,c,\beta}^{\alpha,\gamma}(x))^2\right) + \omega\left(f, \mu_{n,c,\beta}^{\alpha,\gamma}(x)\right).$$

In view of (3.7), we get

$$| \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x) | \quad \leq \quad M\omega_2\left(f, \zeta_{n,c,\beta}^{\alpha,\gamma}(x)\right) + \omega\left(f, \mu_{n,c,\beta}^{\alpha,\gamma}(x)\right),$$

which proves the theorem. Q.E.D.

3.2 Rate of convergence

Let $\omega_a(f,\delta)$ denote the usual modulus of continuity of f on the closed interval $[0,a], a > 0$, and defined as

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,a]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_B[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero. Now, we give a rate of convergence theorem for modified Jain-Baskakov-Stancu operators.

Theorem 3. Let $f \in C_B[0,\infty)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then, for every $n > 3c$,

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x)| \leq 4M_f(1+a^2)\xi_{n,c,\beta}^{\alpha,\gamma}(x) + 2\omega_{a+1}\left(f, \sqrt{\xi_{n,c,\beta}^{\alpha,\gamma}(x)}\right),$$

where $\xi_{n,c,\beta}^{\alpha,\gamma}(x)$ is defined in Remark 2.4 and M_f is a constant depending only on f.

Proof. For $x \in [0,a]$ and $t > a+1$. Since $t - x > 1$, we have

$$\begin{aligned}
|f(t) - f(x)| &\leq M_f(2+t^2+x^2) \\
&\leq M_f(t-x)^2(2+2x+2x^2) \\
&\leq 4M_f(1+a^2)(t-x)^2.
\end{aligned}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \delta>0.$$

From the above, we have

$$|f(t)-f(x)| \leq 4M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),$$

for $x \in [0,a]$ and $t \geq 0$. Thus

$$\begin{aligned}|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| &\leq 4M_f(1+a^2)(\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(t-x)^2;x)\\ &\quad +\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}(\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(t-x)^2;x)^{\frac{1}{2}}\right)\end{aligned}$$

Applying Cauchy-Schwarz's inequality, we get

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| \leq 4M_f(1+a^2)\xi_{n,c,\beta}^{\alpha,\gamma}(x)+2\omega_{a+1}\left(f,\sqrt{\xi_{n,c,\beta}^{\alpha,\gamma}(x)}\right),$$

on choosing $\delta=\sqrt{\xi_{n,c,\beta}^{\alpha,\gamma}(x)}$. This completes the proof of theorem. Q.E.D.

3.3 Weighted approximation.

Let C_ρ be the space of all continuous functions on $[0,\infty)$ with the norm $\| f \|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ and $C_\rho^0=\{f \in C_\rho : \lim_{x\to\infty}\frac{|f(x)|}{\rho(x)}<\infty\}$, where $\rho(x)$ is a weight function.
In what follows we consider $\rho(x)=1+x^2$.

Theorem 4. If $f \in C_\rho^0$, $\lim_{n\to\infty}\beta_n=0$ and $n>3c$, we have

$$\lim_{n\to\infty} \| \mathcal{K}_{n,c,\beta_n}^{\alpha,\gamma}(f)-f \|_\rho=0.$$

Proof. From [3], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| \mathcal{K}_{n,c,\beta_n}^{\alpha,\gamma}(t^r;x)-x^r \|_\rho=0,\ r=0,1,2. \tag{3.12}$$

Since $\mathcal{K}_{n,c,\beta_n}^{\alpha,\gamma}(1;x)=1$, the condition in (3.12) holds for $r=0$.
For $n>2c$, we have

$$\begin{aligned}\| \mathcal{K}_{n,c,\beta_n}^{\alpha,\gamma}(t;x)-x \|_\rho &= \sup_{x\in[0,\infty)}\frac{|\mathcal{K}_{n,c,\beta_n}^{\alpha,\gamma}(t;x)-x|}{1+x^2}\\ &\leq \left|\frac{n^2}{(n+\gamma)(n-2c)(1-\beta_n)}-1\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\left|\frac{\alpha}{n+\gamma}\right|\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\end{aligned}$$

which implies that $\lim_{n\to\infty} \| \mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(t;x) - x \|_\rho = 0$ with $\beta_n \to 0$.
Similarly, we can write for $n > 3c$

$$\begin{aligned}\| \mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(t^2;x) \quad - \quad x^2 \|_\rho &= \sup_{x\in[0,\infty)} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(t^2;x) - x^2|}{1+x^2} \\ &\leq \left|\frac{n^4}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta_n)^2} - 1\right| \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\ &\quad + \left|\frac{n^3 + 2\alpha n^2(n-3c)(1-\beta_n)^2}{(n-2c)(n-3c)(n+\gamma)^2(1-\beta_n)^3}\right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha^2}{(n+\beta_n)^2} \sup_{x\in[0,\infty)} \frac{1}{1+x^2},\end{aligned}$$

which implies that $\lim_{n\to\infty} \| \mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(t^2;x) - x^2 \|_\rho = 0$ with $\beta_n \to 0$.
This completes the proof of theorem. Q.E.D.

Now we give the following theorem to approximate all functions in C^0_ρ. Such type of results are given in [4] for locally integrable functions.

Theorem 5. Let $\beta_n \to 0$ as $n \to \infty$. For each $f \in C^0_\rho$ and $\vartheta > 0$, we have

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f;x) - f(x)|}{(1+x^2)^{1+\vartheta}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\sup_{x\in[0,\infty)} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f;x) - f(x)|}{(1+x^2)^{1+\vartheta}} \leq \sup_{x\leq x_0} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f;x) - f(x)|}{(1+x^2)^{1+\vartheta}} + \sup_{x\geq x_0} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f;x) - f(x)|}{(1+x^2)^{1+\vartheta}}$$

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f;x) - f(x)|}{(1+x^2)^{1+\vartheta}} &\leq \| \mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(f) - f \|_{C[0,x_0]} \\ &\quad + \| f \|_\rho \sup_{x\geq x_0} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(1+t^2;x)|}{(1+x^2)^{1+\vartheta}} + \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\vartheta}}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 3. By Lemma 2.2, for any fixed $x_0 > 0$, it is easily prove that

$$\sup_{x\geq x_0} \frac{|\mathcal{K}^{\alpha,\gamma}_{n,c,\beta_n}(1+t^2;x)|}{(1+x^2)^{1+\vartheta}} \to 0$$

as $n \to \infty$ with $\beta_n \to 0$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be small.
Hence the proof is completed. Q.E.D.

3.4 Pointwise estimates

In this section, we establish some pointwise estimates of the rate of convergence of the operators $\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}$. First, we give the relationship between the local smoothness of f and local approximation. We know that a function $f \in C[0,\infty)$ is in $\text{Lip}_M(\eta)$ on E, $\eta \in (0,1]$, E$\subset [0,\infty)$ if it satisfies the condition

$$|f(t)-f(x)| \leq M|t-x|^{\eta},\ t\in[0,\infty)\ \ and\ \ x\in E,$$

where M is a constant depending only on η and f.

Theorem 6. Let $f \in C[0,\infty)\cap Lip_M(\eta)$, $E\subset[0,\infty)$ and $\eta\in(0,1]$. Then, we have

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| \ \ \leq\ \ M\bigg(\Big(\xi_{n,c,\beta}^{\alpha,\gamma}(x)\Big)^{\eta/2}+2d^{\eta}(x,E)\bigg),\ \ x\in[0,\infty),$$

where M is a constant depending on η and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E)=\inf\{|t-x|:t\in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $x_0\in\overline{E}$ such that

$$d(x,E)=|x-x_0|.$$

By our hypothesis and the monotonicity of $\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}$, we get

$$\begin{aligned}|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| &\leq \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|f(t)-f(x_0)|;x)+\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|f(x)-f(x_0)|;x)\\ &\leq M\left(\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x_0|^{\eta};x)+|x-x_0|^{\eta}\right)\\ &\leq M\left(\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x|^{\eta};x)+2|x-x_0|^{\eta}\right).\end{aligned}$$

Now, applying Hölder's inequality with $p=\dfrac{2}{\eta}$ and $q=\dfrac{2}{2-\eta}$, we obtain

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}((f;x)-f(x)|\leq M\left(\{\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x|^2;x)\}^{\eta/2}+2d^{\eta}(x,E)\right),$$

from which the desired result immediate. Q.E.D.

Next, we obtain the local direct estimate of the operators defined in (1.4), using the Lipschitz-type maximal function of order η introduced by B. Lenze [12] as

$$\widetilde{\omega}_{\eta}(f,x)=\sup_{t\neq x,\ \ t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^{\eta}},\ \ x\in[0,\infty)\ \text{ and }\ \eta\in(0,1]. \tag{3.13}$$

Theorem 7. Let $f\in C_B[0,\infty)$ and $0<\eta\leq1$. Then, for all $x\in[0,\infty)$ we have

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)|\leq\widetilde{\omega}_{\eta}(f,x)\left(\xi_{n,c,\beta}^{\alpha,\gamma}(x)\right)^{\eta/2}.$$

Proof. From the equation (3.13), we have

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)|\leq \widetilde{\omega}_{\eta}(f,x)\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x|^{\eta};x).$$

Applying the Hölder's inequality with $p=\dfrac{2}{\eta}$ and $q=\dfrac{2}{2-\eta}$, we get

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)|\leq \widetilde{\omega}_{\eta}(f,x)\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}((t-x)^2;x)^{\frac{\eta}{2}}\leq \widetilde{\omega}_{\eta}(f,x)\left(\xi_{n,c,\beta}^{\alpha,\gamma}(x)\right)^{\eta/2}.$$

Thus, the proof is completed. Q.E.D.

Let us consider the Lipschitz-type space with two parameters [25]:
For $a,b>0$, we define

$$Lip_M^{(a,b)}(\eta)=\left(f\in C[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^{\eta}}{(t+ax^2+bx)^{\eta/2}};\ x,t\in[0,\infty)\right),$$

where M is any positive constant and $0<\eta\leq 1$.

Theorem 8. For $f\in Lip_M^{(a,b)}(\eta)$. Then, for all $x>0$, we have

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)|\leq M\left(\frac{\xi_{n,c,\beta}^{\alpha,\gamma}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Proof. First we prove the theorem for $\eta=1$. Then, for $f\in Lip_M^{(a,b)}(1)$, and $x\in[0,\infty)$, we have

$$\begin{aligned}|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| &\leq \mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|f(t)-f(x)|;x)\\ &\leq M\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}\left(\frac{|t-x|}{(t+ax^2+bx)^{1/2}};x\right)\\ &\leq \frac{M}{(ax^2+bx)^{1/2}}\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x|;x).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| &\leq \frac{M}{(ax^2+bx)^{1/2}}\left(\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}((t-x)^2;x)\right)^{1/2}\\ &\leq M\left(\frac{\xi_{n,c,\beta}^{\alpha,\gamma}(x)}{ax^2+bx}\right)^{1/2}.\end{aligned}$$

Thus the result holds for $\eta=1$.
Now, we prove that the result is true for $0<\eta<1$. Then, for $f\in Lip_M^{(a,b)}(\eta)$, and $x\in[0,\infty)$, we get

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| \leq \frac{M}{(ax^2+bx)^{\eta/2}}\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x|^{\eta};x).$$

Taking $p=\frac{1}{\eta}$ and $q=\frac{2}{2-\eta}$, applying the Hölders inequality, we have

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| \leq \frac{M}{(ax^2+bx)^{\eta/2}}\left(\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(|t-x|;x)\right)^{\eta}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)-f(x)| \quad \leq \quad M\left(\frac{\xi_{n,c,\beta}^{\alpha,\gamma}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Thus, the proof is completed. Q.E.D.

4 King's approach

To make the convergence faster, King [11] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t)=t^i, i=0,1,2$. After this approach many researcher contributed in this direction.
As the operator $\mathcal{K}_{n,c,\beta}^{\alpha,\gamma}(f;x)$ defined in (1.4) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions.
For this purpose the modification of (1.4) is defined as

$$\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x) \quad = \quad \frac{(n-c)}{n}\sum_{v=1}^{\infty}\omega_\beta(v,nr_n(x))\int_0^\infty p_{n,v-1,c}(t)f\left(\frac{nt+\alpha}{n+\gamma}\right)dt+e^{-nr_n(x)}f\left(\frac{\alpha}{n+\gamma}\right) \tag{4.1}$$

where $r_n(x)=\frac{(n-2c)(1-\beta)((n+\gamma)x-\alpha)}{n^2}$ and $x\in I_n=[\frac{\alpha}{n+\gamma},\infty)$.

Lemma 4.1. For every $x\in I_n$, we have

1. $\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(1;x)=1$;
2. $\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(t;x)=x$;
3. $\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(t^2;x)=\frac{(n-2c)}{(n-3c)}x^2+\frac{n-2c\alpha(1-\beta)^2}{(n-3c)(n+\gamma)(1-\beta)^2}x+\frac{c\alpha^2(1-\beta)^2-n\alpha}{(n-3c)(n+\gamma)^2(1-\beta)^2}$.

Consequently, for each $x\in I_n$, we have the following equalities

$$\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(t-x;x)=0$$

$$\begin{aligned}\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}((t-x)^2;x) &= \frac{c}{(n-3c)}x^2+\frac{n-2c\alpha(1-\beta)^2}{(n-3c)(n+\gamma)(1-\beta)^2}x+\frac{c\alpha^2(1-\beta)^2-n\alpha}{(n-3c)(n+\gamma)^2(1-\beta)^2}\\ &= \lambda_{n,c,\beta}^{\alpha,\gamma}(x), \text{(say)}.\end{aligned} \tag{4.2}$$

Theorem 9. For $f \in C_B(I_n)$ and $n > 3c$, we have

$$|\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x)| \leq M'\omega_2\left(f, \sqrt{\lambda_{n,c,\beta}^{\alpha,\gamma}(x)}\right),$$

where $\lambda_{n,c,\beta}^{\alpha,\gamma}(x)$ is given by (4.2) and M' is a positive constant.

Proof. Let $g \in W^2$ and $x, t \in I_n$. Using the Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}$ on both sides and using Lemma 4.1, we get

$$\hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(g;x) - g(x) = \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}\left(\int_x^t (t-v)g''(v)dv; x\right).$$

Obviously, we have $\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2\|g''\|$.
Therefore

$$| \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(g;x) - g(x) | \leq \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}((t-x)^2;x) \parallel g'' \parallel = \lambda_{n,c,\beta}^{\alpha,\gamma}(x) \parallel g'' \parallel.$$

Since $| \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x) | \leq \|f\|$, we get

$$\begin{aligned} | \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x) | &\leq | \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f-g;x) | + | (f-g)(x) | + | \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(g;x) - g(x) | \\ &\leq 2\|f-g\| + \lambda_{n,c,\beta}^{\alpha,\gamma}(x)\|g''\|. \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$ and using (3.7) we obtain

$$| \hat{\mathcal{K}}_{n,c,\beta}^{\alpha,\gamma}(f;x) - f(x) | \leq M'\omega_2\left(f, \sqrt{\lambda_{n,c,\beta}^{\alpha,\gamma}(x)}\right),$$

which proves the theorem. Q.E.D.

Theorem 10. Let $b > 0$ and $\beta_n \in (0,1)$ such that $n\beta_n \to l \in R$ as $n \to \infty$. Then for every $f \in C_B\left[\frac{\alpha}{n+\gamma}, b\right]$, f', f'' exists at a fixed point $x \in \left(\dfrac{\alpha}{n+\gamma}, b\right)$, we have

$$\lim_{n\to\infty} n\left(\hat{\mathcal{K}}_{n,c,\beta_n}^{\alpha,\gamma}(f;x) - f(x)\right) = \frac{x(1+cx)}{2}f''(x).$$

The proof follows along the lines of Theorem 1.

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